KRANNERT SCHOOL OF MANAGEMENT

Purdue University
West Lafayette, Indiana

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By

Mohit Tawarmalani
Jean-Philippe P. Richard

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Mohit Tawarmalani · Jean-Philippe P. Richard

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Abstract In this paper, we consider decomposition techniques in the convexification of inequalities. We demonstrate these techniques by convexifying an inequality between two monomials.

Theorem 1 *Consider a continuous function* $f : \mathbb{R}^n \mapsto \mathbb{R}$ *and a convex set* $X \subseteq R^n$*. Define*

$$T = \{x \mid f(x) = 0, x \in X\},$$
$$T^{\geq} = \{x \mid f(x) \geq 0, x \in X\}, \text{ and } T^{\leq} = \{x \mid f(x) \leq 0, x \in X\}.$$

Then, $\text{conv}(T) = \text{conv}(T^{\geq}) \cap \text{conv}(T^{\leq})$.

Proof Since $T = T^{\geq} \cap T^{\leq} \subseteq \text{conv}(T^{\geq}) \cap \text{conv}(T^{\leq})$ and $\text{conv}(T^{\geq}) \cap \text{conv}(T^{\leq})$ is convex, it follows that $\text{conv}(T) \subseteq \text{conv}(T^{\geq}) \cap \text{conv}(T^{\leq})$. Now, to show the reverse inclusion by deriving a contradiction, we assume that there exists an $x' \in \text{conv}(T^{\geq}) \cap \text{conv}(T^{\leq})$ that does not belong to $\text{conv}(T)$. We perform

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M. Tawarmalani
Krannert School of Management, Purdue University, USA,
E-mail: mtawarma@purdue.edu

J.-P. P. Richard
Department of Industrial and Systems Engineering, University of Florida, USA,
E-mail: richard@ise.ufl.edu

an induction on the dimension of X. The result is clearly true if X is zero-dimensional. By separating hyperplane theorem, there exists a hyperplane H such that $T \subseteq \text{conv}(T) \subseteq H^-$ and $x' \in H^+$, where H^- and H^+ are opposing closed half-spaces associated with H. Further, it can be arranged that T is not contained in H. We show that either $T^{\geq} \not\subseteq H^-$ or $T^{\leq} \not\subseteq H^-$. Assume otherwise. This implies that $x' \in H$, $\text{conv}(T \cap H) = \text{conv}(T) \cap H$, $\text{conv}(T^{\geq} \cap H) = \text{conv}(T^{\geq}) \cap H$, and $\text{conv}(T^{\leq} \cap H) = \text{conv}(T^{\leq}) \cap H$. Therefore, $x' \in \text{conv}(T^{\geq} \cap H) \cap \text{conv}(T^{\leq} \cap H)$ and $x' \notin \text{conv}(T \cap H)$. However, this is contradictory to the induction hypothesis because $X \backslash H \supseteq T \backslash H \neq \emptyset$ implies that $X \cap H$ has a strictly smaller dimension than X.

We may therefore assume that there exists an $\hat{x} \in T^{\geq} \cap H^+$ and $\bar{x} \in T^{\leq} \cap H^+$ neither of which belongs to T and at least one of these points does not belong to H. Therefore, $f(\hat{x}) > 0$ and $f(\bar{x}) < 0$. Since f is continuous, there exists a point x'' along the line segment $(\hat{x}, \bar{x})$ such that $f(x'') = 0$. Since X is convex, $x'' \in X$. Therefore, $x'' \in T \backslash H^-$. However, this yields a contradiction to the construction of the separating hyperplane which guarantees that $T \subseteq H^-$. In other words, $\text{conv}(T) = \text{conv}(T^{\geq}) \cap \text{conv}(T^{\leq})$. □

1 Convex Hull of Inequalities: Motivation

In this section we consider the set:

$$G = \left\{ (x, y) \in \mathbb{R}^n_+ \times [L, U] \;\middle|\; \prod_{i=1}^{n} (x_i)^{b_i} \geq y \right\} \tag{1}$$

Proposition 1 *Let $b_i \geq 0$ for $i = 1, \ldots, n$ and $0 < L < U$. If $\sum_{i=1}^n b_i > 1$, then*

$$\text{conv}(G) = \left\{ (x, y) \in \mathbb{R}^n_+ \times [L, U] \;\middle|\; \prod_{i=1}^{n} (x_i)^{b_i} \geq \frac{U - y}{U - L} L^{\frac{1}{\sum_{i=1}^n b_i}} + \frac{y - L}{U - L} U^{\frac{1}{\sum_{i=1}^n b_i}} \right\}. \tag{2}$$

Otherwise, $\text{conv}(G) = G$.

When $\sum_{i=1}^n b_i > 1$, the inequality defining (2) is clearly valid. This is because G may be rewritten as:

$$G = \left\{(x,y) \in \mathbb{R}_+^n \times [L,U] \;\middle|\; \prod_{i=1}^n (x_i)^{\frac{b_i}{\sum_{i=1}^n b_i}} \geq y^{\frac{1}{\sum_{i=1}^n b_i}}\right\}, \tag{3}$$

where the left-hand-side and right-hand-side are concave functions when $\sum_{i=1}^n b_i > 1$ and $b_i \geq 0$ for $i = 1,\ldots,n$. It follows that (2) is valid since $y^{\frac{1}{\sum_{i=1}^n b_i}} \geq \frac{U-y}{U-L} L^{\frac{1}{\sum_{i=1}^n b_i}} + \frac{y-L}{U-L} U^{\frac{1}{\sum_{i=1}^n b_i}}$. The defining inequality is a generalization of the inequality derived using (**??**).

Proof The last statement is obvious since the defining inequality is convex when $\sum_{i=1}^n b_i \leq 1$. We assume $t \geq 0$. Let $\phi_\alpha(t) = \inf\{\alpha^T x \mid \prod_{i=1}^n (x_i)^{b_i} \geq t\}$. Wlog we assume that $b_i > 0$ for all i. Now, consider the case where $\alpha_j < 0$. Then, by considering $x_j = \gamma t^{\frac{1}{b_i}}$ and $x_i = 1$ for $i \neq j$, it follows that $\phi_\alpha(t) = -\infty$. Therefore, we restrict attention to $\alpha \geq 0$. Since the objective function is non-decreasing in x, there exists an optimal solution $\bar{x}$ that satisfies $\prod_{i=1}^n (\bar{x}_i)^{b_i} = t$. If for any j, $\alpha_j = 0$, then consider a feasible solution x. Observe that for any $\gamma > 0$, x' is a feasible solution where $x'_i = \gamma x_i$ for $j \neq i$ and $x'_j = x_j \left(\frac{1}{\gamma}\right)^{\frac{\sum_{i \neq j} b_i}{b_j}}$. Then, by taking γ down towards zero, it follows that $\phi_\alpha(t) = 0$ for all $t \geq 0$. Now, we consider $\alpha < 0$. By optimality conditions, there exists a λ such that $\bar{x}_j = \lambda \frac{b_j}{\alpha_j}$. Let $B = \sum_{i=1}^n b_i$ and $a_i = \frac{b_i}{B}$. Then, by substituting in $\prod_{i=1}^n (\bar{x}_i)^{b_i} = t$, it follows that $\bar{x}_j = t^{\frac{1}{B}} \frac{b_j}{\alpha_j} \prod_{i=1}^n \left(\frac{\alpha_i}{b_i}\right)^{a_i}$. Further, $\phi_\alpha(t) = t^{\frac{1}{B}} \prod_{i=1}^n \left(\frac{\alpha_i}{b_i}\right)^{a_i} \sum_{i=1}^n b_i$. Clearly, $\phi_\alpha(t)$ is concave because $B > 1$. Then, using the argument at the beginning of Section **??**, an inequality is valid for $\operatorname{conv}(G)$ if it is of the form:

$$(U-L)\alpha^T x - B\left((U-y)L^{\frac{1}{B}} + (y-L)U^{\frac{1}{B}}\right) \prod_{i=1}^n \left(\frac{\alpha_i}{b_i}\right)^{a_i} \geq 0.$$

Now, we compute, for a given x, the α at which the left-hand-side is minimized. Let $p_i = (U-L)x_i$ and $s = B\left((U-y)L^{\frac{1}{B}} + (y-L)U^{\frac{1}{B}}\right) \prod_{i=1}^n \left(\frac{1}{b_i}\right)^{\frac{1}{a_i}}$. Observe that $p \geq 0$ and $s \geq 0$ over the region of interest. Then, the convexification problem reduces to $\inf_{\alpha \geq 0}\{\alpha^T p - s \prod_{i=1}^n (\alpha_i)^{a_i}\} \geq 0$. Although the value function of this problem is well-known, we include its short derivation for completeness. Since the objective is homogenous, we may impose a normalization:

$$\inf_{\alpha \geq 0} \{\alpha^T p - s \prod_{i=1}^{n} (\alpha_i)^{a_i} \mid \alpha^T p = 1\} \geq 0. \tag{4}$$

It is easy to see, following a similar argument to the one used in the computation of ϕ, that the optimal solution to this problem is $\alpha_i = \frac{a_i}{p_i}$. Then, (4) reduces to:

$$s \prod_{i=1}^{n} (a_i)^{a_i} \leq \prod_{i=1}^{n} (p_i)^{a_i} \tag{5}$$

Substituting the expressions for p_i and s as defined above, yields the defining inequality of the right-hand-side of (2). Alongwith the bounds on y, (5) gives $\text{conv}(G)$. □

Porposition 1 is interesting in the context of recent work on relaxations for convex-transformable functions [4]. First, we recall some basic definitions. A function ϕ is called G-concave if G is a continuous real-valued increasing function and $G(\phi(x))$ is concave [1]. If ϕ is G_*-concave, then G_* is the least concavifying function of G if for every G such that ϕ is G-concave, GG_*^{-1} is concave [1]. Least concavifying functions for concavifiable monomials were derived in [4] and these constructions were applied to develop convex relaxations of $\phi(x) \geq t$. The procedure simply rewrites the defining inequality as $G(\phi(x)) \geq G(t)$, exploiting that G is increasing, and then relaxes $G(t)$ over the relavant domain using a convex underestimator, say $\bar{G}(t)$. The use of G_* is preferred over any other G, because $G(\phi(x)) \geq \bar{G}(t)$ may be rewritten as $G_* G^{-1} \circ G(\phi(x)) \geq G_* G^{-1} \circ \bar{G}(t)$ revealing that the convex underestimator $G_* G^{-1} \circ \bar{G}$ of G_* yields the same relaxation [4]. Observe that the construction of G_* is in general not obvious and such a function often does not exist because ϕ may not be concavifiable. Nevertheless, if G_* exists, one wonders whether $\{(x,t) \in \mathbb{R}^n \times [L,U] \mid G_*(\phi(x)) \geq \text{conv}(G_*)(t)\}$ yields the convex hull of $\{(x,t) \in \mathbb{R}^n \times [L,U] \mid \phi(x) \geq t\}$, thereby generalizing Proposition 1. However, we show in the next example that such is not the case even for univariate functions.

Example 1 Consider the function

$$\phi(x) = \begin{cases} -\frac{1}{2}x^3 + 85x^2 + 1850x + 9500 & x \geq 0 \\ -\infty & \text{otherwise} \end{cases} \tag{6}$$

and the set $P = \{(x,t) \in \mathbb{R}_+ \times [13536, 36000] \mid \phi(x) \geq t\}$. First, observe that the function is convex when $0 \leq x \leq \frac{170}{3}$ and concave when $\frac{170}{3} \leq x$. We claim that when $\phi(x)$ is composed with

$$G = \begin{cases} \frac{1}{300}t + \frac{20}{3} & t \leq 1000 \\ t^{\frac{1}{3}} & \text{otherwise,} \end{cases} \tag{7}$$

the resulting function is concave. It is easy to verify that $\frac{dG(\phi(x))}{dx}$ is decreasing for $0 \leq x \leq \frac{170}{3}$ because the function is twice-differentiable in this region and the second derivative is negative. Also, $\frac{dG(\phi(x))}{dx}$ is decreasing when $x > \frac{170}{3}$ because ϕ is concave in this region and G is an increasing concave function. Therefore, we conclude that $G(\phi(x))$ is concave. Since G is concavifiable, there exists a G_* [2]. In our argument, we will however not construct G_* explicitly. Instead, we show a stronger result. We consider any two transformations, $H : \mathbb{R} \mapsto \mathbb{R}$ and $\bar{H} : \mathbb{R} \mapsto \mathbb{R}$ that are such that $F = \{(x,t) \mid H(\phi(x)) \geq \bar{H}(t)\}$ is a convex superset of P. We then show that $F \supsetneq \text{conv}(P)$. In particular, observe that we do not require H to be increasing, $H(\phi(x))$ to be concave, or $\bar{H}$ to be a convex relaxation of H. It is easy to check that $(2, 13536)$ and $(10, 36000)$ belong to P. Therefore, by convexity of F and because F contains P, we conclude that $(6, 24768) \in F$. Observe that, $\phi(82 + 4\sqrt{713}) = \phi(6) = 23552$. Therefore, $H(\phi(82 + 4\sqrt{713})) = H(\phi(6)) \geq \bar{H}(24678)$, where the inequality follows since $(6, 24768) \in F$. Therefore, $(82+4\sqrt{713}, 24768) \in F$. Now, observe that the function

$$\phi'(x) = \begin{cases} 16 \times 10^6 & 0 \leq x \leq \frac{370}{3} \\ -\frac{1}{2}x^3 + 85x^2 + 1850x + 9500 & x > \frac{370}{3} \end{cases} \tag{8}$$

is concave and $\phi'(x) \geq \phi(x)$. Therefore, $\text{conv}(P) \subseteq X = \{(x,t) \in \mathbb{R}_+ \times [13536, 36000] \mid \phi'(x) \geq t\}$. But $(82+4\sqrt{713}, 24768) \notin X$. Therefore, $\text{conv}(P) \subsetneq$

F. As a special case, it follows that the use of least concavifying function does not necessarily yield the convex hull of P. Also, observe that $X \supseteq T = \{(x,t) \in \mathbb{R}_+ \times [13536, 36000] \mid \text{conc}_{\mathbb{R}_+}(\phi)(x) \geq t\}$, where $\text{conc}_{\mathbb{R}_+}(\cdot)$ denotes the concave envelope over $\mathbb{R}_+$. Therefore, $F \subsetneq T$. This reveals that concavifying transformations do not always yield the best relaxations because they maintain level sets whereas convex envelopes can alter the level sets. We have also shown that, in general, it is not sufficient to separately transform the left-hand-side and the right-hand-side of an inequality in order to build its convex hull even over fairly simple regions. □

Consider $G(\phi(x)) \geq \text{conv}(G)(t)$, where G is strictly increasing and continuous and $\text{conv}(G)(\cdot)$ is a convex envelope of G over $[L, U]$. Observe that, if $-\infty < L \leq U < \infty$, then $\text{conv}(G)$ is strictly increasing. First observe that, in this case $\text{conv}(G)$ is continuous. Now, if for $t_1 < t_2$, $\text{conv}(G)(t_1) \geq \text{conv}(G)(t_2)$, then there exists a $t \geq t_2 > t_1$ such that $G(t) \leq \text{conv}(G)(t_1) \leq G(t_1)$, which is impossible because G is strictly increasing. If the slope of $\text{conv}(G)$ is finite at L and U, then one can extend it so that the domain of the function is $\mathbb{R}$. Then, one can view the above relaxation in the conventional setting as $\text{conv}(G)^{-1} \circ G(\phi(x)) \geq t$ and observe that $\text{conv}(G)^{-1} \circ G$ is a concavifying operator that overestimates $\phi(x)$ in $\{x \mid \phi(x) \in [L, U]\}$.

2 Perspective Decomposition

We will show now that there is a natural generalization of Proposition 1. Let the horizon cone of a set A be defined as:

$$\mathcal{H}A = \left\{ \lambda x \;\middle|\; \lambda \geq 0, \exists x^\nu \in A, \|x^\nu\| \to \infty, \frac{x^\nu}{\|x^\nu\|} \to x \right\}.$$

Theorem 2 *Consider $A \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^n$, $R \subseteq \mathbb{R}^m$, and $z \in \mathbb{R}$. We assume that A is nonempty, $R \subseteq \{(z,r) \mid z \geq 0\}$, and that there exists $(z', r') \in R$ where $z' > 0$. Define $S = \{(x, z, r) \mid x \in zA \text{ if } z > 0; x \in T \text{ if } z = 0; (z, r) \in R\}$. and*

$T \subseteq 0^+ \operatorname{cl}\operatorname{conv}(A)$. *In particular,* T *may be* $\mathcal{H}A$. *Define*

$$A'(z) = \begin{cases} z \operatorname{cl}\operatorname{conv}(A) & \textit{if } z > 0 \\ 0^+ \operatorname{cl}\operatorname{conv}(A) & \textit{if } z = 0 \end{cases}$$

$$C = \{(x, z, r) \mid (z, r) \in \operatorname{cl}\operatorname{conv}(R), x \in A'(z)\},$$

Then, $C = \operatorname{cl}\operatorname{conv}(S)$.

Proof It suffices to show that $S \subseteq C \subseteq \operatorname{cl}\operatorname{conv}(S)$ and that C is a closed convex set. First, we show that $S \subseteq C$. Since $T \subseteq 0^+ \operatorname{cl}\operatorname{conv}(A)$, the inclusion is obvious. In order to show that T may be taken to be $\mathcal{H}A$, we need to show that $\mathcal{H}A \subseteq 0^+ \operatorname{cl}\operatorname{conv}(A)$. Let $x \in \mathcal{H}A$. We show that $\bar{x} \in 0^+ \operatorname{cl}\operatorname{conv}(A)$. Since both sets are positively homogenous, we may additionally assume that $\|x\| = 1$. Then, there exists $x^\nu \in A$, such that $\|x^\nu\| \to \infty$ and $\frac{x^\nu}{\|x^\nu\|} \to x$. Let $y \in A$. It follows that for any $\lambda > 0$ there exists ν large enough such that $\|x^\nu\| \geq \lambda$. Then,

$$y + \frac{\lambda}{\|x^\nu\|}(x^\nu - y) \to y + \lambda x \in \operatorname{cl}\operatorname{conv}(A).$$

Therefore, $x \in 0^+ \operatorname{cl}\operatorname{conv}(A)$.

Now, we show the second inclusion, *i.e.*, $C \subseteq \operatorname{cl}\operatorname{conv}(S)$. Let $(\bar{x}, \bar{z}, \bar{r}) \in C$. First, we assume that $\bar{z} > 0$. We argue that it suffices to show that $C' = \{(x, z, r) \mid x \in z \operatorname{conv}(A), (z, r) \in \operatorname{conv}(R)\} \subseteq \operatorname{cl}\operatorname{conv}(S)$. Let $(\bar{x}, \bar{z}, \bar{r}) \in C \backslash C'$. Then, there exist $(z^\nu, r^\nu) \in \operatorname{conv}(R)$ such that $(z^\nu, r^\nu) \to (\bar{z}, \bar{r})$ and $y^\nu \to y$ such that $y = \frac{\bar{x}}{\bar{z}}$. But then, $y^\nu \bar{z}^\nu \to \bar{x}$. Therefore, $(\bar{x}, \bar{z}, \bar{r}) \in \operatorname{cl} C'$. Then, $C \subseteq \operatorname{cl} C' \subseteq \operatorname{cl}\operatorname{conv}(S)$, where the last inclusion follows from the closedness of $\operatorname{cl}\operatorname{conv}(S)$. We may therefore assume that $(\bar{x}, \bar{z}, \bar{r}) \in C'$. Now, let Δ_k denote the simplex $\{\lambda \mid \sum_{i=1}^k \lambda_i = 1, \lambda_i \geq 0\}$. Since $(\bar{z}, \bar{r}) \in \operatorname{conv}(R)$, there exist $(z^i, r^i) \in R$ and $\gamma^i \in \Delta_{m+1}$ such that $(\bar{z}, \bar{r}) = \sum_{i=1}^{m+1} \gamma_i (z^i, r^i)$. Further, there exist y^j in A, $\lambda_j \in \Delta_{n+1}$ such that $\bar{x} = \bar{z} \sum_{i=1}^{n+1} y^j \lambda_j$. Then, it is easy to see that $\operatorname{vec}(\lambda \gamma^T) \in \Delta_{(n+1)(m+1)}$ and $(\bar{x}, \bar{z}, \bar{r}) = \sum_{j=1}^{n+1} \sum_{i=1}^{m+1} \lambda_j \gamma_i (z^i y^j, z^i, r^i)$. When $z^i > 0$, $(z^i y^j, z^i, r^i) \in S$. Otherwise, $(0, 0, r^i) \in S$. Therefore, $(\bar{x}, \bar{z}, \bar{r}) \in \operatorname{cl}\operatorname{conv}(S)$.

Now, consider $\bar{z} = 0$. Then, $\bar{x} \in 0^+ \operatorname{cl}\operatorname{conv}(A)$ and $(0, \bar{r}) \in \operatorname{cl}\operatorname{conv}(R)$. Since A is nonempty, and $(z', r') \in R$, it follows that there exists a $y \in \operatorname{cl}\operatorname{conv}(A)$ such that $(yz', z', r') \in \operatorname{cl}\operatorname{conv}(S)$. For $\gamma \in (0, 1)$, $\gamma(0, \bar{r}) + (1 - \gamma)(z', r') \in \operatorname{conv}(R)$. Therefore, $(y(1-\gamma)z', (1-\gamma)z', (1-\gamma)r' + \gamma\bar{r}) \in \operatorname{cl}\operatorname{conv}(S)$. By letting γ approach 1, it follows from the closedness of $\operatorname{cl}\operatorname{conv}(S)$ that $(0, 0, \bar{r}) \in \operatorname{cl}\operatorname{conv}(S)$. Since $\bar{x} \in 0^+ \operatorname{cl}\operatorname{conv}(A)$, it follows that for all $\lambda > 0$, $\Big((y + \frac{1}{z'\lambda}\bar{x})z', z', r'\Big) \in \operatorname{cl}\operatorname{conv}(S)$. Consequently for $\lambda \in (0, 1)$,

$$(0, 0, \bar{r}) + \lambda\big((yz' + \frac{1}{\lambda}\bar{x}, z', r') - (0, 0, \bar{r})\big) \underset{\lambda \to 0}{\to} (\bar{x}, 0, \bar{r}) \in \operatorname{cl}\operatorname{conv}(S).$$

Therefore, $C \subseteq \operatorname{cl}\operatorname{conv}(S)$.

Now, we show that C is a convex set. Let $p^1, p^2 \in C$. We wish to show that $p = (\bar{x}, \bar{z}, r) = \lambda p^1 + (1 - \lambda)p^2 \in C$. First assume that z^1 and z^2 are positive. Then, $p^1 = (y^1 z^1, z^1, r^1)$ and $p^2 = (y^2 z^2, z^2, r^2)$. Then, $y^1, y^2 \in \operatorname{cl}\operatorname{conv}(A)$ and $(z^1, r^1), (z^2, r^2) \in \operatorname{cl}\operatorname{conv}(R)$. By the definition of C, it follows that $(y^1 z^2, z^2, r^2), (y^2 z^1, z^1, r^1) \in C$. Let $\lambda \in (0, 1)$. Define $\theta = \frac{\lambda z^1}{\bar{z}}$. Observe that $0 \leq \theta \leq 1$. Then, it follows that $p = \lambda\theta p^1 + (1 - \lambda)(1 - \theta)p^2 + \lambda(1 - \theta)p^3 + (1 - \lambda)\theta p^4$. By convexity of $\operatorname{cl}\operatorname{conv}(A)$, it follows that $\theta y^1 + (1 - \theta)y^2 \in \operatorname{cl}\operatorname{conv}(A)$. Then, it follows that $\theta p^2 + (1 - \theta)p^4$ and $\theta p^1 + (1 - \theta)p^3$ belong to C. Also, by convexity of $\operatorname{cl}\operatorname{conv}(R)$, $\lambda(z^1, r^1) + (1 - \lambda)(z^2, r^2) \in \operatorname{cl}\operatorname{conv}(R)$. Therefore, $\lambda\theta p^1 + (1 - \lambda)(1 - \theta)p^2 + \lambda(1 - \theta)p^3 + (1 - \lambda)\theta p^4 \in C$. Assume $z^2 = 0$ and $z^1 > 0$. Then, $p^2 = (x^2, 0, r^2)$, where $x^2 \in 0^+ \operatorname{cl}\operatorname{conv}(A)$ and $p^1 = (y^1 z^1, z^1, r^1)$. Clearly, $(\lambda z^1, \lambda r^1 + (1 - \lambda)r^2) \in \operatorname{cl}\operatorname{conv}(R)$. Therefore, $(\bar{x}, \bar{z}, \bar{r}) = ((y^1 + \frac{1-\lambda}{\lambda z^1}x^2)\lambda z^1, \lambda z^1, \lambda r^1 + (1 - \lambda)r^2) \in C$. Now, consider $\bar{z} = 0$. Then, it follows that $z^1 = z^2 = 0$. But, then $p \in C$, because $\operatorname{cl}\operatorname{conv}(R)$ is convex and $0^+ \operatorname{cl}\operatorname{conv}(A)$ is also convex.

We finally show that C is closed. Consider a sequence $(x^\nu, z^\nu, r^\nu) \in C$ that converges to $(\bar{x}, \bar{z}, \bar{r})$. Assume $\bar{z} > 0$. For sufficiently large ν and $0 < \delta < \bar{z}$, we may assume that $z^\nu > \delta$. Now, consider $\frac{x^\nu}{z^\nu}$. Since x^ν converges to $\bar{x}$, for a sufficiently large ν and E, it can be assumed that $\|x^\nu\| < E$. Therefore, $\|y^\nu\| \leq \frac{E}{\delta}$. Since $\operatorname{cl}\operatorname{conv}(A) \cap \{y \mid y \leq \frac{E}{\delta}\}$ is a compact set, by considering a subsequence if necessary, y^ν converges to $\bar{y} \in \operatorname{cl}\operatorname{conv}(A)$. Since $\bar{y}\bar{z} = \bar{x}$ it follows that $(\bar{x}, \bar{z}, \bar{r}) \in C$. Now, assume $\bar{z} = 0$. Consider a sequence $(x^\nu, z^\nu, r^\nu) \in C$ that converges to $(\bar{x}, \bar{z}, \bar{r})$. If there does not exist a large enough N such that for all $\nu \geq N$, $z^\nu > 0$, then by restricting the sequence to the indices with $z^\nu = 0$, it follows that $(\bar{x}, \bar{z}, \bar{r}) \in C$ because the intersection of C with $\{(x, z, r) \mid z = 0\}$ is closed. So, we may assume that $z^\nu > 0$. Therefore, there exists a sequence $y^\nu \in \operatorname{cl}\operatorname{conv}(A)$ such that $y^\nu z^\nu$ converges to $\bar{x}$. We show that $\bar{x} \in 0^+ \operatorname{cl}\operatorname{conv}(A)$. Consider $y' \in \operatorname{cl}\operatorname{conv}(A)$. For any $s > 0$, for sufficiently large ν, $sz^\nu \leq 1$. Therefore, $y' + sz^\nu(y^\nu - y') \to y' + s\bar{x} \in \operatorname{cl}\operatorname{conv}(A)$. □

Corollary 1 *Consider* $B \subseteq \mathbb{R}^n$, $K \subseteq \mathbb{R}^n$, $R \subseteq \mathbb{R}^m$, *and* $z \in \mathbb{R}$. *Let* $S = \{(x, z, r) \mid x \in zB \text{ if } z > 0; x \in K \text{ if } z = 0; (z, r) \in R\}$, *where* K *denotes a cone. We assume that* B *is nonempty,* $R \subseteq \{(z, r) \mid z \geq 0\}$, *there exists* $(z', r') \in R$ *where* $z' \neq 0$, *and* $(0, r'') \in R$. *Let*

$$B'(z) = \begin{cases} z \operatorname{cl}\operatorname{conv}(B + K) & \text{if } z > 0 \\ 0^+ \operatorname{cl}\operatorname{conv}(B + K) & \text{if } z = 0 \end{cases}$$

Define

$$C = \{(x, z, r) \mid (z, r) \in \operatorname{cl}\operatorname{conv}(R), x \in B'(z)\},$$

Then, $C = \operatorname{cl}\operatorname{conv}(S)$.

Proof Let

$$B''(z) = \begin{cases} z(B + K) & \text{if } z > 0 \\ K & \text{if } z = 0 \end{cases}$$

Let $S' = \{(x, z, r) \mid (z, r) \in R, x \in B''(z)\}$. We only need to show that $S' \subseteq \operatorname{cl}\operatorname{conv}(S)$. The rest of the result then follows from Theorem 2 by defining $A = B + K$ and $T = K$. Assume $(\bar{x}, \bar{z}, \bar{r}) \in S'$. If $\bar{z} = 0$, there is nothing to prove since $(\bar{x}, \bar{z}, \bar{r}) \in S$. If $\bar{z} > 0$, there exists $y \in B$ and $d \in K$ such that $\bar{x} = \bar{z}(y + d)$. Since $(0, r'') \in R$, it follows that for all $\lambda > 0$, $(\frac{\bar{z}}{\lambda} d, 0, r'') \in S$. Therefore,

$$(\bar{z}y, \bar{z}, \bar{r}) + \lambda\Big((\frac{\bar{z}}{\lambda} d, 0, r'') - (\bar{z}y, \bar{z}, \bar{r})\Big) \underset{\lambda \to 0}{\to} (\bar{z}(y + d), \bar{z}, \bar{r}) \in \operatorname{cl}\operatorname{conv}(S).$$

Therefore, $S' \subseteq \operatorname{cl}\operatorname{conv}(S)$ and, consequently, $C = \operatorname{cl}\operatorname{conv}(S)$. □

Now, we combine the Theorem 2 and Corollary 1 into the following result.

Corollary 2 *Consider* $A_k \subseteq \mathbb{R}^{n_k}$, $k = 1, \ldots, K$, $T_k \subseteq \mathbb{R}^{n_k}$, $R_0 \subseteq \mathbb{R}^m$, *and* $z \in \mathbb{R}^K$. *Let* $R_0 \subseteq \{(z, r) \mid z \geq 0\}$. *Define* $I = \{k \mid \exists (z^k, r^k) \in R_0 \text{ with } z_k^k = 0\}$ *and* $J = \{k \mid \exists (z^k, r^k) \in R_0 \text{ with } z_k^k > 0\}$. *For* $k \in J$ *assume that* $A_k \neq \emptyset$. *For* $k \in I \cap J$, *assume that* T_k *is a cone. Let* $S_K = \{(x_1, \ldots, x_K, z, r) \mid (z, r) \in R_0, x_k \in z_k A_k \text{ if } z_k > 0; x_k \in T_k \text{ if } z_k = 0\}$. *Define*

$$A'_k(z) = \begin{cases} z_k \operatorname{cl}\operatorname{conv}(A_k + T_k) & \text{if } z > 0 \text{ and } k \in I \cap J \\ 0^+ \operatorname{cl}\operatorname{conv}(A_k + T_k) & \text{if } z = 0 \text{ and } k \in I \cap J \\ z_k \operatorname{cl}\operatorname{conv}(A_k) & \text{if } k \in J \backslash I \\ \operatorname{cl}\operatorname{conv}(T_k) & \text{if } k \in I \backslash J \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$C_K = \{(x_1, \ldots, x_K, z, r) \mid (z, r) \in \operatorname{cl}\operatorname{conv}(R_0), x_k \in A'_k(z), k = 1, \ldots, K\},$$

Then, $C_K = \operatorname{cl}\operatorname{conv}(S_K)$.

Proof We proceed by induction on K, Consider $K = 1$. If $k \in I \cap J$, the result follows from Corollary 1. If $k \in J \cap I$, the result follows from Theorem 2 because $\operatorname{cl}\operatorname{conv}(R_0)$ does not meet $\{(z, r) \mid z = 0\}$. If $k \in I \backslash J$, $\operatorname{cl}\operatorname{conv}(R_0) \subseteq \{(z, r) \mid z = 0\}$. Then, the result follows easily. When $k \notin I \cup J$. $R_0 = \emptyset$ and the result follows trivially. Now, we assume the result is true for $K = p$ and prove for $K = p + 1$. Rewrite S_{p+1} as $S_{p+1} = \{(x_1, \ldots, x_{p+1}, z, r) \mid x_{p+1} \in z_{p+1} A_{p+1}, (z, r, x_1, \ldots, x_p) \in S_p\}$. Then, $\operatorname{cl}\operatorname{conv}(S_p)$ is obtained using the induction hypothesis and the rest of the result is derived just as in case of $K = 1$. □

Next, we treat perspective functions.

Corollary 3 *Let* $f(x) : \mathbb{R}^n \to \mathbb{R}$ *be a lower-semicontinuous convex function. Then,*

$$\operatorname{cl}\operatorname{conv}\left\{(x,y,z) \;\middle|\; \left\{\begin{matrix} f(x) \le y \\ z = 1 \end{matrix}\right\} \cup \left\{\begin{matrix} f0^+(x) \le y \\ z = 0 \end{matrix}\right\}\right\}$$
$$= \left\{(x,y,z) \;\middle|\; zf\left(\frac{x}{z}\right) \le y, 0 \le z \le 1\right\},$$

where $0f\left(\frac{x}{0}\right)$ *is defined to be* $f0^+(x)$*.*

Proof Let $A = \{(y,x) \mid y \ge f(x)\}$ and $R = \{z \mid z \in \{0,1\}\}$. Since f is lower-semicontinuous and convex, $A = \operatorname{cl}\operatorname{conv} A$. Further, by definition of the recession function, $0^+A = \{(y,x) \mid f0^+(x) \le y\}$. It is easy to see that for $z > 0$, $zA = \{(zy, zx) \mid f(x) \le y\} = \{(y,x) \mid f\left(\frac{x}{z}\right) \le \frac{y}{z}\}$. The rest of the result follows from Theorem 2. $\square$

Lemma 1 *Let* $A = \{x \mid f(x) \ge 1\}$*, where* f *is a positively-homogenous function that is continuous over its domain, which is a closed set. Let* $A' = \{x \mid \operatorname{cl}\operatorname{conc}_X(f)(x) \ge 1\}$ *where* $X = \operatorname{cl}\{x \mid f(x) > 0\}$*. Then,* $\operatorname{cl}\operatorname{conv}(A) = A'$*.*

Proof Observe that A' is closed convex and contains A. Therefore, it suffices to show that $A' \subseteq \operatorname{cl}\operatorname{conv}(A)$. Also, observe that $x \in \operatorname{cl}\operatorname{conv}(A)$ if there exists an $\alpha \ge 1$ such that $\frac{x}{\alpha} \in \operatorname{cl}\operatorname{conv}(A)$. Therefore, we only need to show that for any $\bar{x} \in A'$, there exists an $\alpha \ge 1$ such that $\frac{\bar{x}}{\alpha} \in \operatorname{cl}\operatorname{conv}(A)$. Since $\bar{x} \in A'$, there exist $x^i_\nu \in X$, $i = 1, \ldots, n+1$, and $\lambda^i_\nu \in \Delta_{n+1}$ such that $x^\nu = \sum_{i=1}^{n+1} \lambda^i_\nu x^i_\nu$, $x_\nu \to \bar{x}$, and $\lim_{\nu\to\infty} \sum_{i=1}^{n+1} \lambda^i_\nu f(x^i_\nu) \ge 1$. Without loss of generality, we assume that $\lambda^i_\nu > 0$ for all i. Since f is continuous, it follows that $f(x) \ge 0$ for all $x \in X$. Let $I_\nu = \{i \mid f(x^i_\nu) > 0\}$. For $i \in I_\nu$, we let $\bar{y}^i_\nu = x^i_\nu$. If $i \notin I_\nu$, we choose $\bar{y}^i_\nu \in X$ such that $f(\bar{y}^i_\nu) > 0$ and $\|\bar{y}^i_\nu - x^i_\nu\| \le \frac{1}{\nu}$. Observe that $f(\bar{y}^i_\nu) \ge f(x^i_\nu)$. Then, define, for all i, $y^i_\nu = \frac{\bar{y}^i_\nu}{f(\bar{y}^i_\nu)}$. By positive-homogeneity of f, $y^i_\nu \in A$ for all i. Define $\Lambda_\nu = \sum_{i=1}^{n+1} \lambda^i_\nu f(\bar{y}^i_\nu)$ and $\gamma^i_\nu = \frac{\lambda^i_\nu f(\bar{y}^i_\nu)}{\Lambda_\nu}$. Clearly, $\gamma_\nu \in \Delta_{n+1}$. Further, $\sum_{i=1}^{n+1} \gamma^i_\nu f(y^i_\nu) = 1$. Therefore, $p_\nu = \sum_{i=1}^{n+1} \gamma^i_\nu y^i_\nu \in \operatorname{conv}(A)$. However, $p_\nu = \frac{1}{\Lambda_\nu} \sum_{i=1}^{n+1} \lambda^i_\nu \bar{y}^i_\nu$. By construction of $\bar{y}^i_\nu$, it follows that $p_\nu = \frac{1}{\Lambda_\nu} \sum_{i=1}^{n+1} \lambda^i_\nu x^i_\nu + \frac{1}{\Lambda_\nu} \sum_{i \notin I_\nu} \lambda^i_\nu (\bar{y}^i_\nu - x^i_\nu)$. By letting ν

approach ∞ and since $\alpha = \lim_{\nu\to 1} \Lambda_\nu \geq \lim_{\nu\to\infty} \sum_{i=1}^{n+1} \lambda_\nu^i f(x_\nu^i) \geq 1$, it follows that $p_\nu \to \frac{\bar{x}}{\alpha}$. Therefore, $\bar{x} \in \operatorname{cl}\operatorname{conv}(A)$. □

Lemma 2 *Consider $A = \{x \mid f(x) \geq 1\}$ and $K = \{x \mid f(x) \geq 0\}$, where f is a positively-homogenous function that is continuous over its domain, which is a closed set. Let $A' = \{x \mid \operatorname{cl}\operatorname{conc}_K(f)(x) \geq 1\}$. Then,* $\operatorname{cl}\operatorname{conv}(A+K) = A'$.

Proof First, we show that $\operatorname{cl}\operatorname{conv}(A+K) \subseteq A'$. Consider $\bar{x} \in A+K$. Then, there exist x' and x'' such that $\bar{x} = x' + x''$ where $f(x') \geq 1$ and $f(x'') \geq 0$. Since $x' + \frac{1}{\lambda}(\lambda x'' - x') = x' + x''$, it follows that $\operatorname{conc}_K(f)(x'+x'') \geq (1-\frac{1}{\lambda})f(x') + \frac{1}{\lambda}f(\lambda x'') \geq (1-\frac{1}{\lambda})f(x')$, where the inequality follows from the positive homogeneity of f and $f(x'') \geq 0$. Taking λ to infinity, it follows that $\operatorname{cl}\operatorname{conc}_K(f)(x'+x'') \geq 1$. Since A' is closed and convex, it follows that $\operatorname{cl}\operatorname{conv}(A+K) \subseteq A'$. Now we show that $A' \subseteq \operatorname{cl}\operatorname{conv}(A+K)$. Consider $\bar{x} \in A'$. Since $\bar{x} \in A'$, there exist $x_\nu^i \in X$, $i = 1, \ldots, n+1$, and $\lambda_\nu^i \in \Delta_{n+1}$ such that $x_\nu = \sum_{i=1}^{n+1} \lambda_\nu^i x_\nu^i$, $x_\nu \to \bar{x}$, and $\lim_{\nu\to\infty} \sum_{i=1}^{n+1} \lambda_\nu^i f(x_\nu^i) \geq 1$. Let $I_\nu = \{i \mid f(x_\nu^i) > 0\}$. For $i \in I_\nu$, let $y_\nu^i = \frac{x_\nu^i}{f(x_\nu^i)}$ and observe that $f(y_\nu^i) = 1$. Therefore, $y_\nu^i \in A$. Define $\Lambda_\nu = \sum_{i=1}^{n+1} \lambda_\nu^i f(x_\nu^i)$. For $i \in I_\nu$, let $\gamma_\nu^i = \frac{\lambda_\nu^i f(x_\nu^i)}{\Lambda_\nu}$. Since $\gamma_\nu \in \Delta_{|I_\nu|}$ and $y_\nu^i \in A$, it follows that $\sum_{i\in I_\nu} \gamma_\nu^i y_\nu^i = \frac{1}{\Lambda_\nu}\sum_{i\in I_\nu} \lambda_\nu^i x_\nu^i \in \operatorname{conv}(A)$. Further, $\frac{1}{\Lambda_\nu}\sum_{i\notin I_\nu} \lambda_\nu^i x_\nu^i \in \operatorname{conv}(K)$. Therefore, $\frac{1}{\Lambda_\nu} x_\nu \in \operatorname{conv}(A) + \operatorname{conv}(K) = \operatorname{conv}(A+K)$. Taking ν to infinity, and letting $\alpha = \lim_{\nu\to\infty} \Lambda_\nu$, it follows that $\frac{\bar{x}}{\alpha} \in \operatorname{cl}\operatorname{conv}(A+K)$. Realizing that $\alpha \geq 1$, $\bar{x} \in \operatorname{cl}\operatorname{conv}(A+K)$. □

Theorem 3 *Let $S = \{(x,y) \mid f(x) \geq g(y), y \in Y\}$ where $f(x)$ is a positively-homegenous function that is continuous over its domain, which is closed. Assume further that $0 \leq g(\cdot) < \infty$ for all $y \in Y$. Let $A = \{x \mid f(x) \geq 1\}$ and assume that $A \neq \emptyset$. Let $K = \{x \mid f(x) \geq 0\}$ and $X = \operatorname{cl}\{x \mid f(x) > 0\}$. Define*

$$C = \begin{cases} \{(x,y) \mid \operatorname{cl}\operatorname{conc}_K(f)(x) \geq 0, y \in Y\} & \text{if } g(y) = 0 \forall y \in Y \\ \{(x,y) \mid \operatorname{cl}\operatorname{conc}_X(f)(x) \geq \operatorname{cl}\operatorname{conv}_Y(g)(y), y \in Y\} & \text{if } g(y) > 0 \forall y \in Y \\ \{(x,y) \mid \operatorname{cl}\operatorname{conc}_K(f)(x) \geq \operatorname{cl}\operatorname{conv}_Y(g)(y), y \in Y\} & \text{otherwise.} \end{cases}$$

Then, $C = \operatorname{cl}\operatorname{conv}(S)$.

Proof First, assume that $g(y) = 0$ for all $y \in Y$. Then, the result follows easily since $\operatorname{cl}\operatorname{conc}_K(f)(x) \geq 0$ if and only if $x \in \operatorname{cl}\operatorname{conv}(K)$. Define S' by lifting S into a higher dimensional space by introducing z as follows:

$$S' = \{(x, z, r) \mid f(x) \geq z \geq g(y), y \in Y\}.$$

Define $R = \{(z, y) \mid z \geq g(y), y \in Y\}$. Also, it is easy to see that, for any cone T, $\operatorname{cl}\operatorname{conc}_T(f)(x)$ is a positively-homogenous function. If $g(y) > 0$ for all $y \in Y$, then the result follows from Theorem 2 and Lemma 1 by observing that

$$\operatorname{cl}\operatorname{conv}(S') = \{(x, z, r) \mid \operatorname{cl}\operatorname{conc}_X(f) \geq z \geq \operatorname{cl}\operatorname{conv}_Y(g)(y), y \in Y\}.$$

Then, the result follow easily by eliminating z using Fourier-Motzkin elimination. Otherwise, the result follows in a similar manner using Corollary 1 and Lemma 2. □

Observe that Proposition 1 is a special case of Theorem 3. In that case $X = K = \mathbb{R}^n_+$ and $f(x_1, \ldots, x_n) = \prod_{i=1}^n x^{\alpha_i}$, where $\alpha_i = \frac{b_i}{\sum_{i=1}^n b_i}$. Further, since $f(x)$ is concave, we only need to convexify $y^{\frac{1}{\sum_{i=1}^n b_i}}$ which is concave when $\sum_{i=1}^n b_i \geq 1$. Then, using the linear underestimator, which defines the convex envelope over $[L, U]$, Proposition 1 follows. The formula on the right-hand-side of the defining inequality for $\operatorname{conv}(G)$ is precisely this linear underestimator. In fact the following result reveals why least-concavifying transformations may sometimes reveal the convex hull of such inequalities.

Proposition 2 *Consider a function $g(y) : \mathbb{R}^n_+ \to \mathbb{R}$. If there exists a concavifying transformation, f that is increasing, such that $f(g(y))$ is concave and positively-homogeneous then f must be the least concavifying transformation.*

Proof Consider any direction $d \in \mathbb{R}^n_+$ such that $f(g(d)) \neq 0$. This must exist if $g(y)$ is not identically zero since f is an increasing function. Now, consider the function $h(t) = g(td)$ for $t \geq 0$. Then, $f(h(t))$ is concave and positively-homogenous. Therefore, $f(h(t)) = at$ for some $a \neq 0$. Then, it follows that $h(t) = f^{-1}(at)$. Now consider any other transformation $w(\cdot)$ such that $w(g(y))$

is concave. Then, so is $w(h(t)) = w(f^{-1}(at))$. In other words, wf^{-1} is concave. Therefore, f is the least concavifying transformation. □

In Propositions 11 and 12 of [4], it is shown that for all the monomial functions that are concavifiable over the non-negative orthant, where variables with negative powers are strictly positive, there exists an increasing transformation that renders the function concave and positively-homogenous. It follows easily from Proposition 2 that these transformations are least-concavifying as was also shown in Propositions 11 and 12 of [4] using a direct proof. In all these cases, it follows easily from Theorem 3 that the convex hull of an inequality $g(y) \le t$ is obtained easily by considering $f(g(y)) \le \operatorname{cl}\operatorname{conv} f(t)$.

3 Application to Monomial Inequalities

Proposition 3 *Let $b \ge 1$ and consider the function $g(y) = \prod_{i=1}^n y_i^b$. Let $l_i \ge 0$ for $i = 1, \dots, n$ and $H = \prod_{i=1}^n [l_i, +\infty]$. Let $I = \{i \mid l_i > 0\}$. Let*

$$h(y) = \left(\prod_{i\in I} l_i\right)^b \left(1 + \sum_{i\in I}\Big(\frac{y_i}{l_i} - 1\Big)\right)^b.$$

Then,

$$\operatorname{conv}_H g(y) = \begin{cases} h(y) & I = \{1, \dots, n\} \\ \left(y_j \prod_{i\ne j} l_i\right)^b & j \notin I. \end{cases}$$

In particular, $\operatorname{conv}_H g(y) = 0$ *if* $|I| \le n-2$.

Proof First, we consider the case when $I \ne \{1, \dots, n\}$. Assume without loss of generality that $\{1\} \notin I$. Observe that for any $\lambda > 0$ and $z \in \mathbb{R}^n_+$ such that $z_1 = 0$, $\operatorname{cl}\operatorname{conv}_H g(l + \lambda z) = 0$ and therefore by Theorem 8.5 in [6] it follows that $(\operatorname{cl}\operatorname{conv}_H g)0^+(z) = 0$. In particular, for any $y \in H$, it follows that $(\operatorname{cl}\operatorname{conv}_H g)0^+(0, y_2 - l_2, \dots, y_n - l_n) = 0$ and, consequently, $\operatorname{cl}\operatorname{conv}_H g(y) \le \operatorname{cl}\operatorname{conv}_H g(y_1, l_2, \dots, l_n)$. So,

$$\begin{aligned}\operatorname{cl}\operatorname{conv}_H g(y) \le \operatorname{cl}\operatorname{conv}_H g(y_1, l_2, \dots, l_n) &\le g(y_1, l_2, \dots, l_n) \\ &= \left(y_1 \prod_{i=2}^n l_i\right)^b \le \operatorname{cl}\operatorname{conv}_H g(y),\end{aligned}$$

where the first inequality follows from previous discussion, the second inequality since $\operatorname{conv}_H g$ underestimates g, the first equality by the definition of g, and the last inequality because $\left(y_1 \prod_{i=2}^n l_i\right)^b$ is continuous and convex and g is an increasing function of $y_2, \ldots, y_n$. Therefore, equality holds throughout.

Now, we assume that $I = \{1, \ldots, n\}$. We may also assume that $l_i = 1$ for all i. This is because the general case can be obtained from this special case by setting $y = l \circ z$ for all i. Assume the result is true when $l_i = 1$ for all i. Then, it follows that $g'(y) = g(l \circ z) = \left(\prod_{i=1}^n l_i z_i\right)^b$ and $H' = [1, \infty]^n$. Since the result is assumed to hold when all $l_i = 1$, $\operatorname{conv}_{H'} g'(z) = \Big(1 + \sum_{i=1}^n \Big(z_i - 1\Big)\Big)^n$. Consequently, $\operatorname{conv}_H g(y) = h(y)$.

We now consider the case when $H = [1, \infty]^n$. Further, we may assume that $n \geq 2$, otherwise the result follows trivially. We show that it suffices to consider the points on $T = \bigcup_{j=1}^n T_j$ where $T_j = \{(1, \ldots, 1, y_j, 1, \ldots, 1) \mid y_j \geq 1\}$ for building the convex envelope. Towards that end, consider $y \in H \backslash T$. Without loss of generality assume $y_1 \neq 1$ and $y_2 \neq 1$. Define $y' = (y_1 + y_2 - 1, 1, y_3, \ldots, y_n)$, $y'' = (1, y_1 + y_2 - 1, y_3, \ldots, y_n)$ and $\lambda = \frac{y_1 - 1}{y_1 + y_2 - 2}$. Observe that $y = \lambda y_1 + (1 - \lambda) y_2$. Further,

$$\lambda (y_1' y_2')^b + (1 - \lambda)(y_1'' y_2'')^b \geq \left(\lambda y_1' y_2' + (1 - \lambda) y_1'' y_2''\right)^b = (y_1 + y_2 - 1)^b \geq (y_1 y_2)^b,$$

where the first inequality follows from the convexity of z^b over $z \geq 0$ and the second inequality because $y_1 \geq 1$ and $y_2 \geq 1$ implies $(y_1 - 1)(y_2 - 1) \geq 0$ or $y_1 + y_2 - 1 \geq y_1 y_2$ and $(\cdot)^b$ is a non-decreasing function. Therefore, it suffices to consider the points in T for the construction of the $\operatorname{cl}\operatorname{conv}_H g(y)$.

First, assume that $b = 1$. Then, $h(y)$ is linear and agrees with the function over T. Any convex underestimator for $g(y)$ must be no more than $h(y)$ at any point in H and $h(y)$ is convex. Therefore, $\operatorname{conv}_H g(y) = h(y)$. Now, we consider $b > 1$. Observe that $g(y) \geq 0$. Therefore, it follows that $\operatorname{cl}\operatorname{conv}_H g(y) = g^{**}(y)$. As shown above, we may restrict $g(y)$ to T in computing $g^*(\alpha)$. Therefore, $g^*(\alpha) = \sup_j \Big\{\sum_{i \neq j} \alpha_i + \sup_{y_j \geq 1} \{\alpha_j y_j - y_j^b\}\Big\}$. The optimal solution in the inner optimization problem occurs at:

$$y_j^* = \begin{cases} \left(\frac{\alpha_j}{b}\right)^{\frac{1}{b-1}} & \alpha_j \geq b \\ 1 & \text{otherwise.} \end{cases} \tag{9}$$

Now, consider the biconjugate, $g^{**}(y) = \sup_\alpha\{\alpha^T x - g^*(\alpha)\}$. We first show that for the computation of g^{**}, it suffices to consider $\alpha_i \geq b$ for all i. Assume $\alpha_k < b$. Observe that $\frac{dg^*(\alpha)}{\alpha_k} = 1$ since $\alpha_k < b$. Therefore, for any $y \geq 1$, $\alpha^T y - g^*(\alpha)$ is non-decreasing in α_k for $\alpha_k \leq b$. It follows from (9) that $g^*(\alpha) = \sup_j \left\{\sum_{i\neq j} \alpha_i + \frac{b}{d}\left(\frac{\alpha_j}{b}\right)^d\right\}$, where $d = \frac{b}{b-1}$. Let $w(\alpha_j) = \frac{b}{d}\left(\frac{\alpha_j}{b}\right)^d - \alpha_j$. Observe that $\frac{dw(\alpha_j)}{d\alpha_j} = \left(\frac{\alpha_j}{b}\right)^{d-1} - 1$ which is non-negative since $\alpha_j \geq b$ and $d \geq 1$. Therefore, it follows that $g^*(\alpha) = \sum_{i=1}^n \alpha_i + w(\max_i \alpha_i)$. Now, consider $g^{**}(y) = \sup_{\alpha\geq be}\left\{\alpha^T(y-e) - w(\max_i \alpha_i)\right\}$. We may assume that $\alpha = ae$ for some constant a. Consider an α such that $\alpha \geq be$ and $\alpha \neq ae$ for any a. Then, construct $\alpha' = (\max_i \alpha_i)e$ and observe that $\alpha' \geq be$ and $\alpha'^T(y-e) - w(\max_i \alpha'_i) \geq \alpha^T(y-e) - w(\max_i \alpha_i)$ for any $x \geq 1$. Since α' is of the form ae, we may restrict the optimization to such solutions. Then, $g^{**}(y) = \sup_a \left\{ae^T(y - e + e_i) - \frac{b}{d}\left(\frac{a}{b}\right)^d\right\}$. It is easy to see that the optimal value is attained when $a = b\left(e^T(y - e + e_i)\right)^{\frac{b}{d}}$. Substituting, $g^{**}(y) = \operatorname{cl}\operatorname{conv} g(y) = \left(e^T(y - e + e_i)\right)^b = \left(\sum_{i=1}^n y_i - n + 1\right)^b$.

It remains to show that $\operatorname{cl}\operatorname{conv}_H g(y) = \operatorname{conv}_H g(y)$. First, consider $I = \{1, \ldots, n\}$. We proceed by induction on n. When $n = 1$, it follows easily that $h(y) = \operatorname{cl}\operatorname{conv}_H g(y) \leq \operatorname{conv}_H g(y) \leq g(y) = h(y)$ and, therefore, equality holds throughout. Assume that the result is true for $n = k$ and we need to prove it for $n = k + 1$. Since $\operatorname{conv}_H g(y)$ is proper convex function, it follows that $\operatorname{cl}\operatorname{conv}_H g(y) = \operatorname{conv}_H g(y)$ as long as $y \in \operatorname{int}(H)$. If for some i, y_i is fixed to l_i, the problem reduces to showing that $h(y_1, \ldots, y_{i-1}, l_i, y_{i+1}, \ldots, y_n) = \operatorname{conv}_H g(y_1, \ldots, y_{i-1}, l_i, y_{i+1}, \ldots, y_n)$. But this follows from the induction hypothesis. The case where $\{1\} \in I^c$ follows similarly. □

Proposition 4 *Let $g(x, z) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $h(x, y) = g(x, c^T y + d)$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $z \in \mathbb{R}$. Let $C \subseteq \mathbb{R}^n$ and $K \subseteq \mathbb{R}^m$, and $Y \subseteq \{y \mid$*

$c^T y + d = l\}$. *Assume that* $Y \neq \emptyset$, K *is a cone, and* $c \in K^*$, i.e., $c^T y \geq 0$ *for all* $y \in K$. *Let* $S = C \times (Y + K)$ *and* $S' = C \times [l, \infty]$. *Then,*

$$(\mathrm{conv}_S\, h)(x, y) = (\mathrm{conv}_{S'}\, g)(x, c^T y + d).$$

Proof We first show that $v = \min_y\{c^T y + d \mid y \in Y + K\} = l$. Clearly, $v \leq l$, because we may choose any $y^0 \in Y$ to find a feasible solution to the optimization problem defining v that achieves an objective value of l. Further, any feasible y can be expressed as $y = y^0 + y'$ for some $y^0 \in Y$ and $y' \in K$. Therefore, $c^T y + d = c^T(y^0 + y') + d \geq l$, where the last inequality follows since $y^0 \in Y$, $c \in K^*$, and $y' \in K$. Therefore, $v = l$. Consider any $\alpha > (\mathrm{conv}_S\, h)(x, y)$ and observe that there exists $\gamma \in \Delta_{n+m+1}$, $(x^i, y^i) \in S$, for $i = 1, \ldots, n+m-2$, such that $\sum_i \gamma_i(x^i, y^i) = (x, y)$ and $\alpha \geq \sum_i \gamma_i h(x^i, y^i)$. We show that $(\mathrm{conv}'_S\, g)(x, c^T y + d) \leq \alpha$. Observe that $\sum_i \gamma_i(x^i, c^T y^i + d) = (x, c^T y + d)$ and $(\mathrm{conv}'_S\, g)(x, c^T y + d) \leq \sum_i \gamma_i g(x^i, c^T y^i + d) = \sum_i \gamma_i h(x^i, y^i) \leq \alpha$. Therefore, $(\mathrm{conv}_S\, h)(x, y) \geq (\mathrm{conv}'_S\, g)(x, c^T y + d)$.

We now show that $(\mathrm{conv}_S\, h)(x, y) \leq (\mathrm{conv}'_S\, g)(x, c^T y + d)$. Consider $(x, y) \in S$, $y^0 \in Y$, and let $z = c^T y + d$. Then, by the definition of S', $(x, z) \in S'$. Observe that, for every $\alpha > (\mathrm{conv}_S\, g)(x, z)$, there exists $\lambda \in \Delta_{n+2}$, $\lambda > 0$, $(x^i, z^i) \in S'$, for $i = 1, \ldots, n+2$, such that $\sum_i \lambda_i(x^i, z^i) = (x, z)$ and $\alpha \geq \sum_i \lambda_i g(x^i, z^i)$. We need to show that $\alpha \geq (\mathrm{conv}_S\, h)(x, y)$. We consider two cases. First, assume $z = l$. Then, it follows that $z^i = l$ for all i. Observe that $\sum_i \lambda_i(x^i, y) = (x, y)$. Further, $(\mathrm{conv}_S\, h)(x^i, y) \leq h(x^i, y) = g(x^i, l) = g(x^i, z^i)$. Therefore, it follows that $(\mathrm{conv}_S\, h)(x, y) \leq \sum_i \lambda_i g(x^i, z^i) \leq \alpha$. Now, let $z > l$. Define $u = y - y^0$ and observe that $u \in K$ and $c^T u = c^T(y - y^0) = z - l > 0$. Then, define $y^i = y^0 + ut^i$ where $t^i = \frac{z^i - l}{c^T u}$ and observe that $c^T y^i + d = z^i$ and $y^i \in y^0 + K$. Moreover, $\sum_i \lambda_i(x^i, z^i) = (x, z)$ implies that $\sum_i \lambda_i(c^T y^i + d) = c^T y + d$, which simplifies to $\sum_i \lambda_i t^i = 1$ since $c^T u > 0$. It follows that $\sum_i \lambda_i(x^i, y^i) = (x, y)$ and $(\mathrm{conv}_S\, h)(x, y) \leq \sum_i \lambda_i h(x^i, y^i) = \sum_i \lambda_i g(x^i, z^i) \leq \alpha$. Therefore, $(\mathrm{conv}_S\, h)(x, y) \leq (\mathrm{conv}_{S'}\, g)(x, c^T y + d)$. □

The next example illustrates that, in Proposition 4, the requirement that K is a cone cannot be relaxed.

Example 2 Let $h(x,y) = x(y_1+y_2)^2$ and $S = [1,2]^3$, and $l = \min\{y_1 + y_2 \mid 1 \le y_1 \le 2, 1 \le y_2 \le 2\} = 2$. Consider $g(x,z) = xz^2$ over $S' = [1,2]^3$ and observe that for $(x,y) \in S$, $(x, y_1+y_2) \in S'$. Obviously $(\mathrm{cl\,conv}_S\, h)(x,y) \ge (\mathrm{cl\,conv}_{S'}\, g)(x, y_1+y_2)$ for $(x,y) \in S$ as shown in the first part of the proof of Proposition 4. We will show now that there exist $(x,y) \in S$ such that $(\mathrm{cl\,conv}_S\, h)(x,y) > (\mathrm{cl\,conv}'_S\, g)(x, y_1+y_2)$. First consider $(x,y) = (1.5, 1, 2)$. Since S (resp. S') is compact and h (resp. g) is continuous, it follows that the convex hull of the epigraph of h restricted to S (g restricted to S') is closed. Since (x,y) is on a face of S over which it is linear, it follows that $(\mathrm{cl\,conv}_S\, h)(1.5,1,2) = h(1.5,1,2) = 13.5$. However, $(\mathrm{cl\,conv}_{S'}\, g)(1.5,3) \le \frac{g(1,4)+g(2,2)}{2} = 12$. In fact, since $(\mathrm{cl\,conv}_S\, h)(x,y)$ is continuous, it follows that, for sufficiently small r and any $(x,y) \in S$ such that $\|(x,y) - (1.5,1,2)\| < r$, $(\mathrm{cl\,conv}_S\, h)(x,y) > (\mathrm{cl\,conv}_{S'}\, g)(x, y_1+y_2)$. □

Incidentally, Example 2 gives a counterexample to Remark 2 in [3]. Using Propositions 3 and 4, we obtain the following result. The next result can be derived as a special case of the results in [3] or in [5]. For completeness, we provide a direct proof.

Proposition 5 *Consider $g(z,y) = zy^b$ over $S = [L,U] \times [l,\infty]$, where $L \ge 0$, $L < U$, $l \ge 0$ and $b \ge 1$. Define $q = \frac{1}{b-1}$, and $r = \frac{U-z}{U-L}$. If $b > 1$, let $V = \{(y,r) \mid (U^q - L^q)rl \le L^q(y-l)\}$, otherwise $V = \emptyset$. Then,*

$$\mathrm{conv}_S\, g(z,y) = \begin{cases} y^b \frac{(rLU^{qb} + U(1-r)L^{qb})}{\left(L^q + (U^q - L^q)r\right)^b} & (y,r) \in V \text{ and } r > 0 \\ rL\left(\frac{y-l+lr}{r}\right)^b + Ul^b(1-r) & (y,r) \notin V \text{ and } r > 0 \\ Uy^b & r = 0. \end{cases}$$

If $b > 1$, $\mathrm{conv}_S\, g(z,y) = \mathrm{cl\,conv}_S\, g(z,y)$*. Otherwise,* $\mathrm{cl\,conv}_S\, g(x,y) = yL + lx - Ll$*.*

Proof Since zy^b is concave in z for a fixed y it suffices to restrict attention to $S' = \{L,U\} \times [l,\infty]$. Now, let $(z,y) \in S'$. If $z \in \{L,U\}$, it is easy to see that $\mathrm{cl\,conv}_S\, g(z,y) = zy^b$. We now assume that $L < z < U$. Then, there exists $0 < r < 1$, (L, y') and (U, y''), $y' \geq l$ and $y'' \geq l$, such that $r = \frac{U-z}{U-L}$, $ry' + (1-r)y'' = y$, and $\operatorname{conv}_S g(z,y) = rL(y')^b + (1-r)U(y'')^b$. Let t be such that $y'' = y - rt$. Then it follows that $y' = y + (1-r)t$. Therefore,

$$\begin{aligned} \operatorname{cl}\operatorname*{conv}_S g(z,y) = \min_t \quad & rL(y+(1-r)t)^b + (1-r)U(y-rt)^b) \\ & -\frac{y-l}{1-r} \leq t \leq \frac{y-l}{r}. \end{aligned} \tag{10}$$

If $b = 1$, the optimal solution is $\frac{y-l}{r}$ since the objective is a decreasing function in t. Otherwise, it is easy to check that the optimal solution to the above problem is $t^* = \min\left\{\frac{y(U^q-L^q)}{L^q+(U^q-L^q)r}, \frac{y-l}{r}\right\}$. If $(y,r) \in V$, the first expression in the definition of t^* attains the minimum. Then, substituting the appropriate expression for t^* in the objective function of (10), we obtain the expression in the statement of the proposition.

Note that if $b = 1$, $\operatorname{conv}_S g(z,y)$ is not lower-semicontinuous. In particular, if $r > 0$ and $b \to 1$, $\operatorname{conv}_S g(z,y) = yL + lx - Ll$ which, for $y > l$, converges, as $x \to U$, to a value strictly below $\operatorname{conv}_S g(U,y) = Uy$. Therefore, in this case, $\operatorname{cl}\operatorname{conv}_S g(z,y) = yL + lx - Ll$. On the other hand, if $b > 1$, the function in Proposition 5 is continuous and therefore also describes $\operatorname{cl}\operatorname{conv}_S g(x,y)$. $\square$

Proposition 6 *Consider* $g(z,y) = z\prod_{i=1}^n y_i^b$ *over* $S = [L,U] \times \prod_{i=1}^n [l_i, \infty]$, *where* $b \geq 1$. *Assume that* $L \geq 0$, $U > L$, *and, for each* $i = 1, \ldots, n$, $l_i \geq 0$. *Define* $w(y) = \left(\sum_{i=1}^n \left(y_i \prod_{j\neq i} l_j\right) + (n-1)\prod_{i=1}^n l_i\right)^b$ *and* $l_w = \left(\prod_{i=1}^n l_i\right)^b$. *Let* $q = \frac{1}{b-1}$, *and* $r = \frac{U-z}{U-L}$. *If* $b > 1$, *let* $V = \{(y,r) \mid (U^q - L^q)rl_w \leq L^q(w(y) - l_w)\}$, *otherwise let* $V = \emptyset$. *Then,*

$$\operatorname{conv}_S g(z,y) = \begin{cases} w(y)^b \frac{(rLU^{qb} + U(1-r)L^{qb})}{\left(L^q + (U^q - L^q)r\right)^b} & (y,r) \in V \text{ and } r > 0 \\ rL\left(\frac{w(y) - l_w + l_w r}{r}\right)^b + Ul_w^b(1-r) & (y,r) \notin V \text{ and } r > 0 \\ Uw(y)^b & r = 0. \end{cases}$$

Proof Clearly, $\operatorname{conv}_S(f(z)g(y)) = \operatorname{conv}(\operatorname{conv}_{[L,U]} f(z) \operatorname{conv}_{\prod_{i=1}^n [l_i,\infty]} g(y))$. It follows from Proposition 3 that $\operatorname{conv}_{\prod_{i=1}^n [l_i,\infty]} g(y) = w(y)^b$. Observe that $\prod_{i=1}^n [l_i, \infty] = \{l\} + \mathbb{R}_+^n$, where $\mathbb{R}_+^n$ is a cone. Further, the coefficient of y_i in $w(y)$ is non-negative. Therefore, $w(y)$ is minimized at $y = l$, where it attains a value of l_w. Therefore, it follows from Proposition 4 and Proposition 5 that the convex envelope of $g(z, y)$ is as given. □

Lemma 3 *Consider a function* $g(y, z) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, *where* $y \in C \subseteq \mathbb{R}^n$ *and* $z \geq l$. *Let* $g(y, z)$ *be concave and differentiable in* z *for a fixed* y. *Define* $d = \inf\left\{\frac{\partial g(y,z)}{\partial z} \mid (y, z) \in C \times [l, \infty]\right\}$. *Let* $h(y, z) = (\operatorname{cl}\operatorname{conv}_C g(\cdot, l))(y) + d(z-l)$. *Then,* $\operatorname{cl}\operatorname{conv}_{C\times[l,\infty]} g(y, z) = h(y, z)$. *If in particular,* $g(y, z) = w(y)u(z)$, *where* $w(y) \geq l_w \geq 0$ *over* C *and* $\bar{s} = \lim_{z\to\infty} u'(z)$ *then* $\operatorname{cl}\operatorname{conv}_{C\times[l,\infty]} g(y, z) = u(l)\operatorname{cl}\operatorname{conv}_C w(y) + l_w\bar{s}(z - l)$.

Proof First, observe that $h(y, z)$ is a lower-semicontinuous convex function. We now show that $h(y, z) \leq g(y, z)$. The result is obvious if $z = l$. Now, consider (y, z) such that $z > l$. Then, it follows from the concavity of $g(y, \cdot)$ that $g(y, l) \leq g(y, z) + \frac{\partial g(y,z)}{\partial z}(l - z) \leq g(y, z) + d(l - z)$. Therefore, $g(y, z) \geq g(y, l) + d(z - l) \geq h(y, z)$. It follows that $h(y, z) \leq \operatorname{cl}\operatorname{conv}_{C\times[l,\infty]} g(y, z)$. Now, we show the reverse inequality. Let $t(y, z) = \operatorname{cl}\operatorname{conv}_{C\times[l,\infty]} g(y, z)$. Consider $(\bar{y}, \bar{z}) \in C \times [l, \infty]$ and let $\bar{d} = \frac{\partial g(\bar{y},z)}{\partial z}$. We show that $t(y, z) \leq t(y, l) + \bar{d}(z - l)$. Then, since $t(y, l) \leq (\operatorname{cl}\operatorname{conv}_C g(\cdot, l))(y)$ and $\bar{d}$ is arbitrarily chosen the result follows. Observe that

$$\begin{aligned} t0^+(0, z) &= \lim_{\lambda\to\infty} \frac{t(\bar{y}, l + \lambda z) - t(\bar{y}, l)}{\lambda} \\ &\leq \lim_{\lambda\to\infty} \frac{g(\bar{y}, l + \lambda z) - t(\bar{y}, l)}{\lambda} \\ &\leq \lim_{\lambda\to\infty} \frac{g(\bar{y}, \bar{z}) + \bar{d}(l + \lambda z - \bar{z}) - t(\bar{y}, l)}{\lambda} \\ &= \bar{d}z \end{aligned}$$

where the first equality follows from Theorem 8.5 in [6] since t is a proper closed convex function, the first inequality since $t(\bar{y}, l + \lambda z) \leq g(\bar{y}, l + \lambda z)$, and the second inequality follows from the concavity of $g(\bar{y}, \cdot)$. Then, it follows from Theorem 8.5 in [6] that $t(y, z) - t(y, l) \leq t0^+(0, z - l) = \bar{d}(z - l)$.

When $g(y, z) = w(y)u(z)$, then $u'(z)$ is non-increasing in z. Therefore, $\inf\{u'(z) \mid z \geq l\} = \lim_{z\to\infty} u'(z) = \bar{s}$. In this case, $\frac{\partial g(y,z)}{\partial z} = w(y)u'(z)$. Further, $\inf\{w(y)u'(z) \mid y \in C, z \geq l\} = \inf_{y\in C} w(y) \inf_{z\geq l} u'(z) = l_w\bar{s}$. □

As an application of Theorem 3 we obtain the following theorem.

Theorem 4 *Let $lu \geq 0$, $lv \geq 0$, $lw \geq 0$, $L \geq 0$, and $L \leq U$. Let $b_i \geq 0$ for $i = 1, \ldots, n$ and $B = \sum_{i=1}^n b_i$, $b > B$, and $0 \leq d_j < B$ for $j = 1, \ldots, p$. Consider*

$$S = \Big\{(y,u,v,w,z) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}^p_+ \times \mathbb{R}^s_+ \times [L,U] \;\Big|\; \prod_{i=1}^n y_i^{b_i} \geq az \prod_{j=1}^m u_j^b \prod_{j=1}^p v_j^{d_j} \prod_{j=1}^s w_j^B + \lambda, \; u \geq lu, v \geq lv, w \geq lw\Big\},$$

Let $\alpha_i = \frac{b_i}{B}$, $\beta = \frac{b}{B}$, and $\gamma_j = \frac{d_j}{B}$,

$$\chi(u) = \left(\sum_{i=1}^m \left(u_i \prod_{j \neq i} lu_j\right) + (m-1)\prod_{i=1}^m lu_i\right)^\beta,$$

$l_\chi = \left(\prod_{i=1}^n lu_i\right)^b$, $r = \frac{U-z}{U-L}$, $q = \frac{1}{\beta-1}$, and $V = \{(u,r) \mid (U^q - L^q) r l_\chi \leq L^q(w(y) - l_w)\}$. Define $X = \{(y,u,v,w,z) \mid \prod_{i=1}^n {y_i}^{\alpha_i} \geq ah(u,v,w,z) + \lambda\}$, where

$$h(u,v,w,z) = t(u,z)\prod_{j=1}^p lv_j^{\gamma_j} \prod_{j=1}^s lw_j + \sum_{k=1}^s L(w_k - lw_k)\prod_{j=1}^m lu_j^\beta \prod_{j=1}^p lv_j^{\gamma_j} \prod_{j \neq k} lw_j.$$

and

$$t(u,z) = \begin{cases} \chi(u)^\beta \frac{(rLU^{q\beta} + U(1-r)L^{q\beta})}{\left(L^q + (U^q - L^q)r\right)^\beta} & (u,r) \in V \text{ and } r > 0 \\ rL\left(\frac{\chi(u) - l_\chi + l_\chi r}{r}\right)^\beta + Ul_\chi^b(1-r) & (u,r) \notin V \text{ and } r > 0 \\ U\chi(u)^\beta & r = 0. \end{cases}$$

Then, $\operatorname{cl}\operatorname{conv}(S) = X$.

Proof Let $C = \{(u,v,w,z) \mid u \geq lu, v \geq lv, w \geq lw, L \leq z \leq U\}$. By Theorem 3, it follows by raising both sides of the defining inequality to the power $\frac{1}{B}$ that $\operatorname{cl}\operatorname{conv}(S) = \{(y,u,v,w,z) \mid \prod_{i=1}^n {y_i}^{\alpha_i} \geq ag(u,v,w,z) + \lambda\}$, where $g(u,v,w,z) = \operatorname{cl}\operatorname{conv}_C\left(z\prod_{j=1}^m u_j^\beta \prod_{j=1}^p v_j^{\gamma_j} \prod_{j=1}^s w_j\right)$. Therefore, we need to show that $g(u,v,w,z) = h(u,v,w,z)$. Define $f(u,z) = z\prod_{j=1}^m u_j^\beta$. It follows from Lemma 3 that

$$\begin{aligned} g(u,v,w,z) = \quad & \prod_{j=1}^m lv_j^{\gamma_j} \prod_{j=1}^s lw_j^B \operatorname{cl}\operatorname{conv}_{u\geq lu, L\leq z\leq U} f(u,z) \\ & + \sum_{k=1}^s L(w_k - lw_k) \prod_{j=1}^m lu_j^\beta \prod_{j=1}^p lv_j^{\gamma_j} \prod_{j\neq k} lw_j \end{aligned}$$

Therefore, we need to show that $t(u,z) = \operatorname{cl}\operatorname{conv}_{u\geq lu, L\leq z\leq U} f(u,z)$. This follows directly from Proposition 6. □

The above result can be viewed as a template for deriving additional similar results where the right-hand-side may be replaced by any function defined over a set for which the convex envelope is known. Many such results are obtained in [7] and the references therein.

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